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Constructing parametric triangular patches with boundary conditions

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Abstract

The problem of constructing a parametric triangular patch to smoothly connect three surface patches is studied. Usually, these surface patches are defined on different parameter spaces. Therefore, it is necessary to define interpolation conditions, with values from the given surface patches, on the boundary of the triangular patch that can ensure smooth transition between different parameter spaces. In this paper we present a new method to define boundary conditions. Boundary conditions defined by the new method have the same parameter space if the three given surface patches can be converted into the same form through affine transformation. Consequently, any of the classic methods for constructing functional triangular patches can be used directly to construct a parametric triangular patch to connect given surface patches with G^1 continuity. The resulting parametric triangular patch preserves precision of the applied classic method.

Keywords: Triangular patch; Parametric interpolation; Determination of interpolation conditions

1. Introduction

Construction of surfaces plays an important role in computer aided geometric design (CAGD), free-form surface modeling and computer graphics (CG). To make the process of constructing complex surfaces simple, piecewise techniques are frequently used, with four-sided and triangular patches being the most popular choices. This paper studies the problem of boundary condition determination in the process of constructing parametric triangular patches to smoothly connect three given surface patches. There are many methods for constructing a triangular patch to connect the three ones. This paper only addresses the problem of constructing a triangular patch to connect three surfaces with any form, i.e. addresses the problem of infinite interpolation on triangles.

The infinite interpolation on triangles was studied by Barnhill et al. [1], and a curved triangular patch that interpolates the boundary conditions with any form was proposed. The triangular patch is constructed using the Boolean sum scheme. Gregory [2] used the convex combination method to construct a triangular patch. The triangular patch is formed by the convex combination of three interpolation operators, each of which satisfies the interpolation conditions on two sides of a triangle. The idea [2] was further extended by Gregory [3] and Charrot et al. [4]. Nielson [5] presented a side-vertex method to construct a curved triangular patch using combination of three interpolation operators, each satisfying the given boundary conditions at a vertex and its opposite side. Hagen [6] extended Nielson's approach to construct geometric patches. These results have been generalized to triangular patches with first and second order geometric continuity [7,8]. The problem of constructing non-four-sided patches including curved triangular patches was also studied in Refs. [9,10]. In Ref. [11] a method to construct a curved triangular patch by combining four interpolation operators, an

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interior interpolation operator and three side-vertex operators [5], is presented. The constructed triangular patch reproduces polynomial surfaces of degree four. Another method proposed recently [12] constructs a triangular patch by a basic approximation operator and an interpolation operator. The constructed triangular patch satisfies C^1 boundary condition and reproduces polynomial surfaces of degree five.

The above-mentioned methods all work on the assumption that the interpolation conditions on the boundary of the triangle are defined on the same parameter space. However, in practice, this is usually not the case. It is therefore necessary to have a method to determine suitable interpolation conditions so that the methods mentioned in Refs. [1–12] can be used directly to construct parametric triangular patches. In Ref. [11], a method is presented to construct the cross-boundary conditions. The constructed cross-boundary conditions have suitable magnitudes, but not suitable directions on the boundary of the triangle. This paper overcomes this problem by presenting a simple but efficient method to construct cross-boundary conditions which have both suitable magnitudes and directions. The combination of the new method and the classic functional triangular patch construction methods [1–12] can be used to construct a G^1 parametric triangular patch to connect three surface patches. The constructed parametric triangular patch has the same interpolation precision as the classic methods [1–12].

2. Problem description

Suppose $\boldsymbol{P}_i(s_i,t_i)=(x_i(s_i,t_i),y_i(s_i,t_i),z_i(s_i,t_i))$, $(0\leqslant s_i,t_i\leqslant 1), i=1,2,3$, are three given surface patches, defined on different s_it_i-parametric planes. The three patches are of any form, and meet in the way as shown in Fig. 1. The goal is to construct a triangular patch $\boldsymbol{P}_{\mathrm{T}}(s,t)$ to connect the three patches $\boldsymbol{P}_i(s_i,t_i), i=1,2,3$, with G^1 continuity. $\boldsymbol{P}_{\mathrm{T}}(s,t)$ and $\boldsymbol{P}_i(s_i,t_i), i=1,2,3$, being G^1 continuous means that they have a common boundary and the normal vectors of them on the common boundary have the same direction.

If these three patches are defined on the same parametric st-plane, then the methods for constructing functional triangular patches can be used directly to construct a parametric triangular patch to connect these patches with C^1 continuity. In most applications of CAGD, CG and related areas, however, these three patches usually are not defined on the same parameter space. In this case, one needs to define C^1 boundary conditions by the three patches so that the constructed parametric triangular patch can smoothly connect these patches with a "visually pleasing shape" suggested by these three patches. After the C^1 boundary conditions are defined, the functional methods of constructing triangular patches can be used to construct parameter triangular patch directly. Because $\boldsymbol{P}_{\mathrm{T}}(s,t)$ and $\boldsymbol{P}_i(s_i,\quad t_i)$, $i=1,2,3$, are defined on different parameter spaces, $\boldsymbol{P}_{\mathrm{T}}(s,t)$, satisfying C^1 boundary conditions, will connect these three patches with G^1 continuity.

Let T be an equilateral triangle with vertices $\boldsymbol{v}_1=(0,0)$, $\boldsymbol{v}_2=(1,0)$ and $\boldsymbol{v}_3=(1/2,\sqrt{3}/2)$ in the st-parametric space, e_i denote the opposite side of $\boldsymbol{v}_i$, and $\boldsymbol{\tau}_i$ be the unit outward normal vector of e_i (Fig. 2). Let $\boldsymbol{\sigma}_1$ denote the unit vector from $\boldsymbol{v}_2$ to $\boldsymbol{v}_3$. $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ are defined similarly. The sides e_i, $i=1,2,3$, can be parameterized as follows:

$$\begin{aligned} \boldsymbol{e}_1(u)&=(1-u)\boldsymbol{v}_2+u\boldsymbol{v}_3,\\ \boldsymbol{e}_2(u)&=(1-u)\boldsymbol{v}_1+u\boldsymbol{v}_3,\quad 0\leqslant u\leqslant 1\\ \boldsymbol{e}_3(u)&=(1-u)\boldsymbol{v}_1+u\boldsymbol{v}_2,\end{aligned}\tag{1}$$

The parametric triangular patch $\boldsymbol{P}_{\mathrm{T}}(s,t)$ to be constructed will be defined on the equilateral triangle T, as shown in Fig. 2. On the three sides of T, the boundary curve and cross-boundary slope conditions given by the three surfaces, $\boldsymbol{P}_i(s_i,\quad t_i)$, $i=1,2,3$ are as follows:

$$\boldsymbol{P}_i(\boldsymbol{e}_i(u)),\qquad \frac{\partial \boldsymbol{P}_i}{\partial s_i}(\boldsymbol{e}_i(u)),\quad i=1,2,3\tag{2}$$

where $\boldsymbol{e}_i(u),\quad i=1,2,3$ are defined in Eq. (1), $\boldsymbol{P}_i(\boldsymbol{e}_i(u))$ and $\frac{\partial \boldsymbol{P}_i}{\partial s_i}(\boldsymbol{e}_i(u))$ denote the boundary value and the cross-boundary slope of $\boldsymbol{P}_i(s_i,t_i)$ on the side e_i, respectively.

Because $\boldsymbol{P}_i(s_i,t_i), i=1,2,3$ are defined on different parameter spaces, and the boundary conditions (2) cannot be used directly to construct the triangular patch $\boldsymbol{P}_{\mathrm{T}}(s,t)$ on T, we will use them to define the new boundary conditions. Let the new boundary conditions be

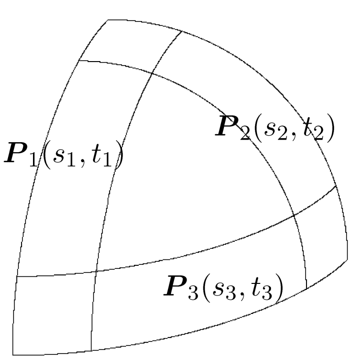

Fig. 1. Three surfaces with G^1 continuity.

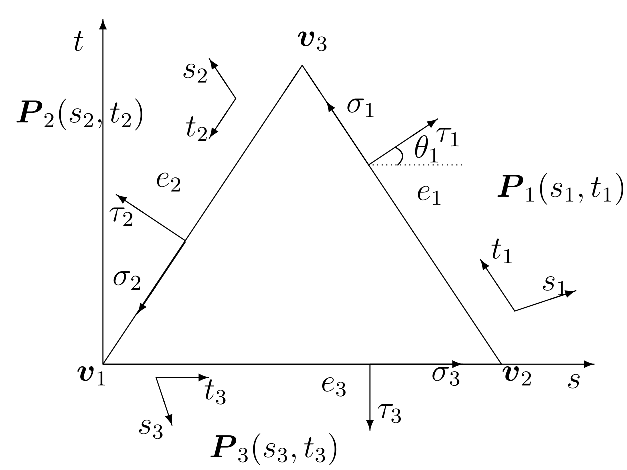

Fig. 2. Three patches on T.

$$\boldsymbol{P}_{\mathrm{T}}(\boldsymbol{e}_i(u)), \qquad \frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_i}(\boldsymbol{e}_i(u)), \quad i=1,2,3 \tag{3}$$

There are many methods for defining the new boundary conditions (3). Obviously, one reasonable choice is that the conditions (3) should be defined in such a way that satisfies the following conditions: if the three patches $\boldsymbol{P}_i(s_i,t_i), \quad i=1,2,3$ are defined by the same surface $\boldsymbol{P}(s,t)$, but with different parameter spaces whose boundary conditions have linear relations between each other, then one should find out the transformations to make sure that the three $\boldsymbol{P}_i(s_i,t_i), \quad i=1,2,3$ are defined by the same surface which is supposed to be $\boldsymbol{P}(s,t)$, so that $\boldsymbol{P}_{\mathrm{T}}(\boldsymbol{e}_i(u))$, $\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_i}(\boldsymbol{e}_i(u)), \quad i=1,2,3$ on the three sides of T in Fig. 2 can be defined by $\boldsymbol{P}(s,t)$, i.e. by

$$\begin{aligned}&\boldsymbol{P}_{\mathrm{T}}(\boldsymbol{e}_i(u))=\boldsymbol{P}(\boldsymbol{e}_i(u)),\\&\tfrac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_i}(\boldsymbol{e}_i(u))=\tfrac{\partial \boldsymbol{P}}{\partial \tau_i}(\boldsymbol{e}_i(u)),\end{aligned} \quad i=1,2,3 \tag{4}$$

3. Constructing the boundary conditions

We now show how to determine $\boldsymbol{P}_{\mathrm{T}}(\boldsymbol{e}_i(u))$, $\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_i}(\boldsymbol{e}_i(u))$, $i=1,2,3$. As shown in Fig. 3, suppose that the surface patch $\boldsymbol{P}_1(s_1,t_1)$ is defined in the parallelogram region $\boldsymbol{v}_2\boldsymbol{v}_3\boldsymbol{v}_4\boldsymbol{v}_5$, $\boldsymbol{P}_2(s_2,t_2)$ and $\boldsymbol{P}_3(s_3,t_3)$ are similarly defined. The $\boldsymbol{P}_{\mathrm{T}}(s,t)$ should be defined so that it and $\boldsymbol{P}_i(s_i, \quad t_i)$ are G^1 continuous on the common boundary. Thus, $\boldsymbol{P}_{\mathrm{T}}(\boldsymbol{e}_i(u))$, $\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_i}(\boldsymbol{e}_i(u)), \quad i=1,2,3$ can be defined by $\boldsymbol{P}_i(s_i,t_i), \quad i=1,2,3$ as follows:

$$\begin{aligned}&\boldsymbol{P}_{\mathrm{T}}(\boldsymbol{e}_i(u))=\boldsymbol{P}_i(\boldsymbol{e}_i(u)),\\&\tfrac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_i}(\boldsymbol{e}_i(u))=\alpha_i(\boldsymbol{e}_i(u))\tfrac{\partial \boldsymbol{P}_i}{\partial s_i}(\boldsymbol{e}_i(u))\\&\quad+\beta_i(\boldsymbol{e}_i(u))\tfrac{\partial \boldsymbol{P}_i}{\partial t_i}(\boldsymbol{e}_i(u)),\end{aligned} \quad i=1,2,3 \tag{5}$$

where $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ are functions of u to be constructed, respectively.

Now, constructing the boundary conditions becomes a problem of defining the functions $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$, $i=1,2,3$. For simplicity, we shall only show the construction process of $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ only. The $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u)), i=2,3$ can be constructed similarly.

Because vectors $\boldsymbol{\sigma}_1$ and $\boldsymbol{t}_1$ are the same (Fig. 3), vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{t}_1$ are orthonormal, and $\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_1}(\boldsymbol{e}_1(u))$ and $\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial t_1}(\boldsymbol{e}_1(u))$ satisfy

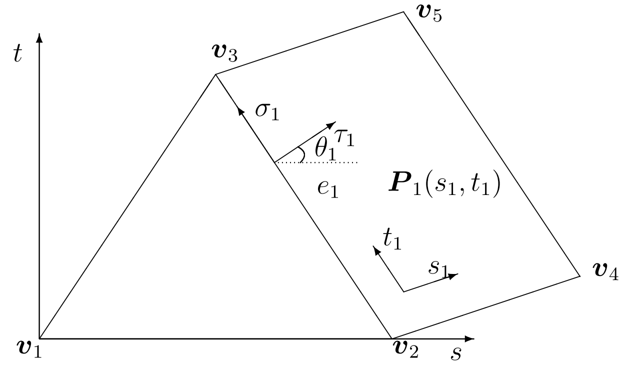

Fig. 3. Three patches on T.

$$\left\langle \frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_1}(\boldsymbol{e}_1(u))\cdot\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial t_1}(\boldsymbol{e}_1(u))\right\rangle=0$$

where $\langle \boldsymbol{a}\cdot\boldsymbol{b}\rangle$ denotes the dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

It follows from (5) that

$$A_1\alpha_1(\boldsymbol{e}_1(u))+B_1\beta_1(\boldsymbol{e}_1(u))=0 \tag{6}$$

where

$$A_1=\left\langle \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{e}_1(u))\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u))\right\rangle$$

$$B_1=\left\langle \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u))\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u))\right\rangle$$

If s_1 and t_1 are orthonormal, $A_1=\left\langle\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{e}_1(u))\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u))\right\rangle=0$, then the function relation between $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ is taken as

$$\beta_1(\boldsymbol{e}_1(u))=-A_1\alpha_1(\boldsymbol{e}_1(u))/B_1 \tag{7}$$

Eq. (7) shows that if $\beta_1(\boldsymbol{e}_1(u))$ is defined, then $\alpha_1(\boldsymbol{e}_1(u))$ is defined. In what follows we will show how to construct $\beta_1(\boldsymbol{e}_1(u))$. We first determine the values of $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ at points $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$, respectively. At point $\boldsymbol{v}_2$, we have

$$\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_1}(\boldsymbol{v}_2)=\alpha_1(v_2)\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)+\beta_1(v_2)\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2) \tag{8}$$

The angle θ_1 between vectors τ_1 and $\boldsymbol{t}_3$ is 30^o, thus

$$\frac{\partial \boldsymbol{P}_3}{\partial t_3}(\boldsymbol{v}_2)=\frac{\sqrt{3}}{2}\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_1}(\boldsymbol{v}_2)-\frac{1}{2}\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \sigma_1}(\boldsymbol{v}_2)$$

From

$$\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \sigma_1}(\boldsymbol{v}_2)=\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)$$

we have

$$\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_1}(\boldsymbol{v}_2)=\frac{2\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_3}{\partial t_3}(\boldsymbol{v}_2)+\frac{\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2) \tag{9}$$

It follows from Eqs. (8) and (9) that $\alpha_1(v_2)$ and $\beta_1(v_2)$ in Eq. (5), denoted by α_1^0 and β_1^0, can be determined by the following equations:

$$\begin{aligned}&\left\langle \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\right\rangle\alpha_1^0+\left\langle \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\right\rangle\beta_1^0\\&\quad=\left\langle \frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\right\rangle\\&\left\langle \frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\right\rangle\alpha_1^0+\left\langle \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\right\rangle\beta_1^0=0\end{aligned} \tag{10}$$

On the other hand, at $\boldsymbol{v}_3$ we have

$$\begin{aligned}&\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_1}(\boldsymbol{v}_3)=\alpha_1(v_3)\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)+\beta_1(v_3)\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\\&\frac{\partial \boldsymbol{P}_{\mathrm{T}}}{\partial \tau_1}(\boldsymbol{v}_3)=-\frac{2\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_2}{\partial t_2}(\boldsymbol{v}_3)-\frac{\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\end{aligned} \tag{11}$$

Thus, $\alpha_1(v_3)$ and $\beta_1(v_3)$ in Eq. (5), denoted by α_1^1 and β_1^1, can be determined by the following equations:

$$\begin{aligned}&\left\langle\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\right\rangle\alpha_1^1+\left\langle\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\right\rangle\beta_1^1\\&\quad=\left\langle\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\right\rangle\\&\left\langle\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\right\rangle\alpha_1^1+\left\langle\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\right\rangle\beta_1^1=0\end{aligned}\tag{12}$$

For $\alpha(\boldsymbol{e}_1(u))$, two values α_1^0 and α_1^1 are computed, thus a suitable choice is that $\alpha_1(\boldsymbol{e}_1(u))$ is defined by a linear interpolation as follows:

$$\alpha_1(\boldsymbol{e}_1(u))=(1-u)\alpha_1^0+u\alpha_1^1\quad 0\leqslant u\leqslant 1\tag{13}$$

where α_1^0 and α_1^1 are defined by (10) and (12).

Based on (7) and (13), $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ are defined by

$$\begin{aligned}&\alpha_1(\boldsymbol{e}_1(u))=(1-u)\alpha_1^0+u\alpha_1^1\\&\beta_1(\boldsymbol{e}_1(u))=-A_1\alpha_1(\boldsymbol{e}_1(u))/B_1\quad 0\leqslant u\leqslant 1\end{aligned}\tag{14}$$

where A_1 and B_1 are defined by (6).

Similarly, one can define $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ for $i=2,3$ as follows:

$$\begin{aligned}&\alpha_2(\boldsymbol{e}_2(u))=(1-u)\alpha_2^0+u\alpha_2^1\\&\beta_2(\boldsymbol{e}_2(u))=-A_2\alpha_2(\boldsymbol{e}_2(u))/B_2\quad 0\leqslant u\leqslant 1\\&\alpha_3(\boldsymbol{e}_3(u))=(1-u)\alpha_3^0+u\alpha_3^1\\&\beta_3(\boldsymbol{e}_3(u))=-A_3\alpha_3(\boldsymbol{e}_3(u))/B_3\end{aligned}\tag{15}$$

The above construction process of C^1 boundary conditions shows that when the methods for constructing C^1 functional triangular patches are directly applied to the boundary conditions in Eq. (5), a parameter patch $\boldsymbol{P}_{\mathrm{T}}(s,t)$ is constructed, which connects $\boldsymbol{P}_i(s_i,t_i)$, $i=1,2,3$ with G^1 continuity and smooth shape.

4. Discussion

In this section, we will show that the cross-boundary slopes defined by Eqs. (5), (14) and (15) are well defined. To do this, we only need to prove that if the three surfaces $\boldsymbol{P}_i(s_i,t_i)$, $i=1,2,3$ are defined by the same surface $\boldsymbol{P}(s,t)$ but in different forms, which are formed by applying affine transformations on $\boldsymbol{P}(s,t)$, then the new boundary conditions are defined by (4), i.e. by $\boldsymbol{P}(s,t)$. This means that if a method reproduces polynomials of degree n when it is used to construct functional triangular patches, then when it is used with the boundary conditions (5) to construct a parametric triangular patch $\boldsymbol{P}_{\mathrm{T}}(s,t)$, $\boldsymbol{P}_{\mathrm{T}}(s,t)$ will reproduce parametric polynomials of degree n.

Theorem 1. *If surface patches $\boldsymbol{P}_i(s_i,t_i)$, $i=1,2,3$, are defined by the same surface $\boldsymbol{P}(s,t)$, i.e. $\boldsymbol{P}(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)$, and the transformations from coordinate system st to coordinate system s_it_i are affine, then there exist unique constants c_i and d_i satisfying the following conditions*:

$$\begin{aligned}&\alpha_i=1/c_i\\&\beta_i=-d_i/c_i\end{aligned}\tag{16}$$

where α_i and β_i satisfy $\alpha_i(\boldsymbol{e}_i(u))=\alpha_i$ and $\beta_i(\boldsymbol{e}_i(u))=\beta_i$, which means that $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ in Eq. (5) are constants in this case.

Proof. Only the case $i=1$ will be considered. The other two cases can be handled similarly. Let $\boldsymbol{V}$ be any point in parametric space, in $\boldsymbol{\tau}_1\boldsymbol{\sigma}_1$ and $\boldsymbol{s}_1\boldsymbol{t}_1$ coordinate systems, the coordinates of $\boldsymbol{V}$ be $(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)$ and $(\boldsymbol{s}_1,\boldsymbol{t}_1)$, respectively. Because the transformation from coordinate system st to coordinate system $\boldsymbol{s}_1\boldsymbol{t}_1$ is affine, vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{t}_1$ are the same, as shown in Fig. 3, the relationship between $(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)$ and $(\boldsymbol{s}_1,\boldsymbol{t}_1)$ can be written as

$$\begin{aligned}&\boldsymbol{\tau}_1=c_1\boldsymbol{s}_1\\&\boldsymbol{\sigma}_1=d_1\boldsymbol{s}_1+\boldsymbol{t}_1\end{aligned}\tag{17}$$

As $\boldsymbol{P}_i(\boldsymbol{s}_1,\boldsymbol{t}_1)$ is defined by $\boldsymbol{P}(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)$, it follows from Eq. (17) that $\boldsymbol{P}_1(\boldsymbol{s}_1,\boldsymbol{t}_1)$ can be expressed as

$$\boldsymbol{P}_1(\boldsymbol{s}_1,\boldsymbol{t}_1)=\boldsymbol{P}(c_1\boldsymbol{s}_1,d_1\boldsymbol{s}_1+\boldsymbol{t}_1)=\boldsymbol{P}(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)$$

Now

$$\begin{aligned}&\frac{\partial\boldsymbol{P}_1(\boldsymbol{s}_1,\boldsymbol{t}_1)}{\partial\boldsymbol{s}_1}=c_1\frac{\partial\boldsymbol{P}(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)}{\partial\boldsymbol{\tau}_1}+d_1\frac{\partial\boldsymbol{P}(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)}{\partial\boldsymbol{\sigma}_1}\\&\frac{\partial\boldsymbol{P}_1(\boldsymbol{s}_1,\boldsymbol{t}_1)}{\partial\boldsymbol{t}_1}=\frac{\partial\boldsymbol{P}(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)}{\partial\boldsymbol{\sigma}_1}\end{aligned}$$

Thus

$$\begin{aligned}&\frac{\partial\boldsymbol{P}(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)}{\partial\boldsymbol{\tau}_1}=\frac{1}{c_1}\frac{\partial\boldsymbol{P}(\boldsymbol{s}_1,\boldsymbol{t}_1)}{\partial\boldsymbol{s}_1}-\frac{d_1}{c_1}\frac{\partial\boldsymbol{P}(\boldsymbol{s}_1,\boldsymbol{t}_1)}{\partial\boldsymbol{t}_1}\\&\frac{\partial\boldsymbol{P}(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)}{\partial\boldsymbol{\sigma}_1}=\frac{\partial\boldsymbol{P}(\boldsymbol{s}_1,\boldsymbol{t}_1)}{\partial\boldsymbol{t}_1}\end{aligned}$$

This completes the proof of the theorem. □

In CAGD and CG applications, the curves and surfaces are generally defined on normalized domains, $[0,1]$ for curves and $[0,1]\times[0,1]$ for surfaces. In most cases, the domains of curves and surfaces are normalized by affine transformations. Thus, in Theorem 1, that the transformation from $\boldsymbol{P}(s,t)$ to $\boldsymbol{P}_i(s_i,t_i)$, $i=1,2,3$ are restricted as affine transformation is reasonable. Theorem 1 shows that if surfaces $\boldsymbol{P}_i(s_i,\ t_i)$, $i=1,2,3$, are defined by the same surface, then α_1^0 and β_1^0 in Eq. (10) and α_1^1 and β_1^1 in Eq. (12) satisfy $\alpha_1^0=\alpha_1^1$ and $\beta_1^0=\beta_1^1$, and so the functions $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ in Eq. (5), $i=1,2,3$ are uniquely determined, i.e. determined by Eq. (4). Consequently, the interpolation conditions are determined uniquely, and the triangular patch to be constructed is determined uniquely. Therefore, the following theorem follows.

Theorem 2. *If the method of constructing functional triangular patch reproduces polynomials of degree n, and the method is directly applied on the interpolation conditions in Eq.* (5), *then the constructed parametric triangular patch $\boldsymbol{P}_{\mathrm{T}}(s,t)$ reproduces parametric polynomials of degree n.*

5. Experiment

Experimental results presented in this section are carried out by constructing a parametric triangular patch to connect three patches or to interpolate the G^1 interpolation conditions on the sides of the triangle. The first experiment is to construct a triangular patch to connect three surfaces, $\boldsymbol{P}_i(s_i, t_i)$, $(0 \leqslant s_i, t_i \leqslant 1)$, $i = 1, 2, 3$ (Fig. 4). The triangular patches in Fig. 5 will be produced by Nielson's method [5]. In Fig. 5, the triangular patch in (a) is produced by directly applying Nielson's method [5] on the boundary curves and cross-boundary slopes defined by the three rectangular patches. The triangular patches in (b) and (c) are produced by using the method presented in Ref. [13] and the technique presented in this paper, respectively, to redefine the cross-boundary slopes taken from the three given rectangular patches, and then apply Nielson's method [5] on the boundary curves and the redefined cross-boundary slopes. In Fig. 5, some portions of the surfaces on the common boundary of the triangular patch with the three rectangular patches are visually not very smooth. This is the result of Mach band phenomenon. The surfaces in (c) have less Mach band phenomenon than those of (b).

Highlight lines [14] have been proved to be an effective tool in assessing the quality of a surface. In Fig. 6, the highlight line model is used to compare the above three methods. Fig. 6 gives the highlight lines of the horizontal fillets of the surfaces in Fig. 5. which shows that the new method gets better results than the other two methods.

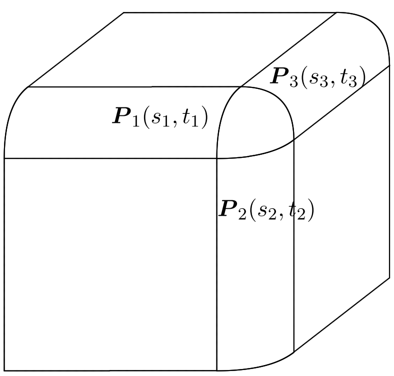

Fig. 4. Connection of three surfaces through a triangular patch.

The second experiment is to compare the new method using the six functions presented by Frank [15]. The six functions are expressed by the following parametric form

$$
\begin{aligned}
F_1(u,v) &= 3.9\exp[-0.25(9u-2)^2 - 0.25(9v-2)^2] \\
&\quad + 3.9\exp[-(9u+1)^2/49 - (9v+1)/10] \\
&\quad + 2.6\exp[-0.25(9u-7)^2 - 0.25(9v-3)^2] \\
&\quad - 1.04\exp[-(9u-4)^2 - (9v-7)^2] \\
F_2(u,v) &= 5.2\exp[18v - 18u]/(9\exp[18v - 18u] + 9) \\
F_3(u,v) &= 5.2[1.25 + \cos(5.4v)]/[6 + 6(3u-1)^2] \\
F_4(u,v) &= 5.2\exp[-81((u-0.5)^2 + (v-0.5)^2)/16]/3 \\
F_5(u,v) &= 5.2\exp[-81((u-0.5)^2 + (v-0.5)^2)/4]/3 \\
F_6(u,v) &= 5.2\mathrm{sqrt}[64 - 81((u-0.5)^2 + (v-0.5)^2)]/9 - 2.6 \\
x(u,v) &= u \\
y(u,v) &= v
\end{aligned}
\tag{18}
$$

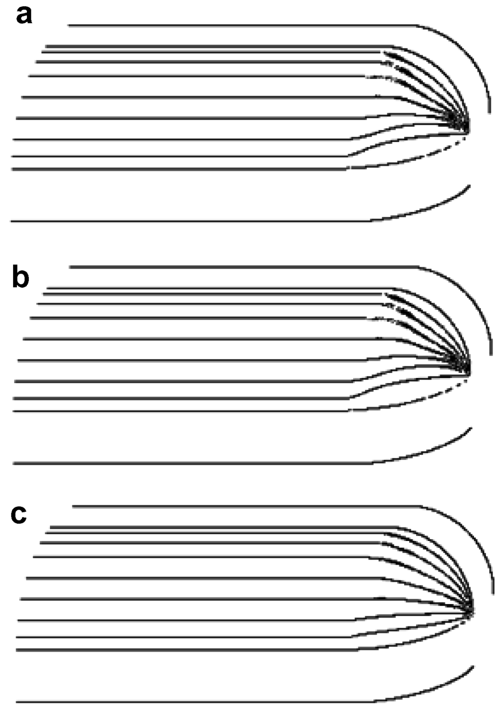

Fig. 6. The highlight lines of the horizontal fillets of the surfaces in Fig. 5.

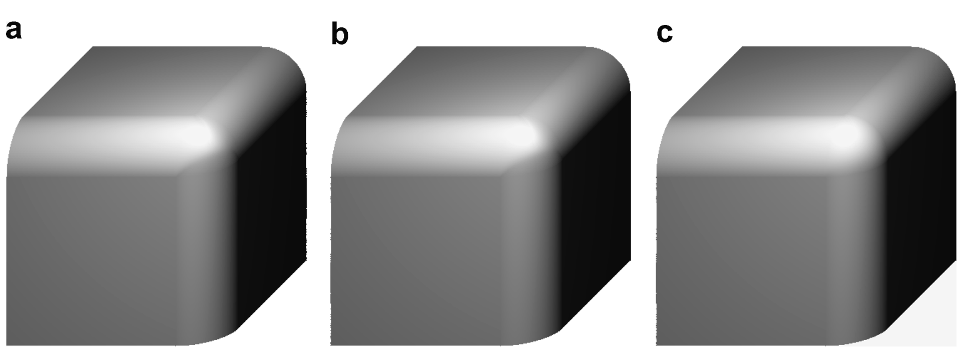

Fig. 5. Triangular patches produced by (a) Nielson's method; (b) Zhang's method and (c) our method.

The set of data points (including 33 points) presented in Ref. [15] is used to produce triangles for comparison. The triangulation of the data set is performed by two steps. First, the data set is projected to xy plane, then the data set on xy plane is triangulated using the max-min criterion proposed by Lawson [16] (Fig. 7). The boundary curves of 3D triangles are defined by $(x, y, F_i(x, y))$, $1 \leqslant i \leqslant 6$, where (x, y) is the point on the side of the triangles in Fig. 7, $F_i(x, y)$ is obtained by replacing (u, v) in $F_i(u, v)$ (18) with (x, y).

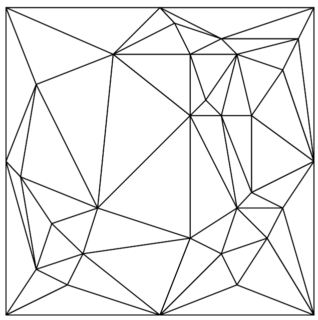

Fig. 7. Triangulation of 33 points.

The interpolation conditions for the test cases are as mentioned boundary curves and cross-boundary slopes on the 3D triangles, taken from $F_1(u, v)$ to $F_6(u, v)$ above. Let S be a side of the 3D triangles, the interpolation conditions on S are normalized by defining them on the unit region, i.e. on the region $[0, 1]$. The cross-boundary slopes on S are defined by $\frac{\partial F_i(u,v)}{\partial \boldsymbol{n}} \times L$, where L denotes the length of S, $\boldsymbol{n}$ denotes the out normal vector of S, $1 \leqslant i \leqslant 6$. Based on the interpolation conditions on the 3D triangles, the comparison is carried out by applying the new method to the method [12] to construct surfaces. The comparison results are shown in Figs. 8–10, respectively. In Figs. 8–10, for $1 \leqslant i \leqslant 6$, the surface (a) $F_i(u, v)$ is produced by using formula (18), the surface (b) $F_i(u, v)$ is produced with the method [12] by directly using the interpolation condi-

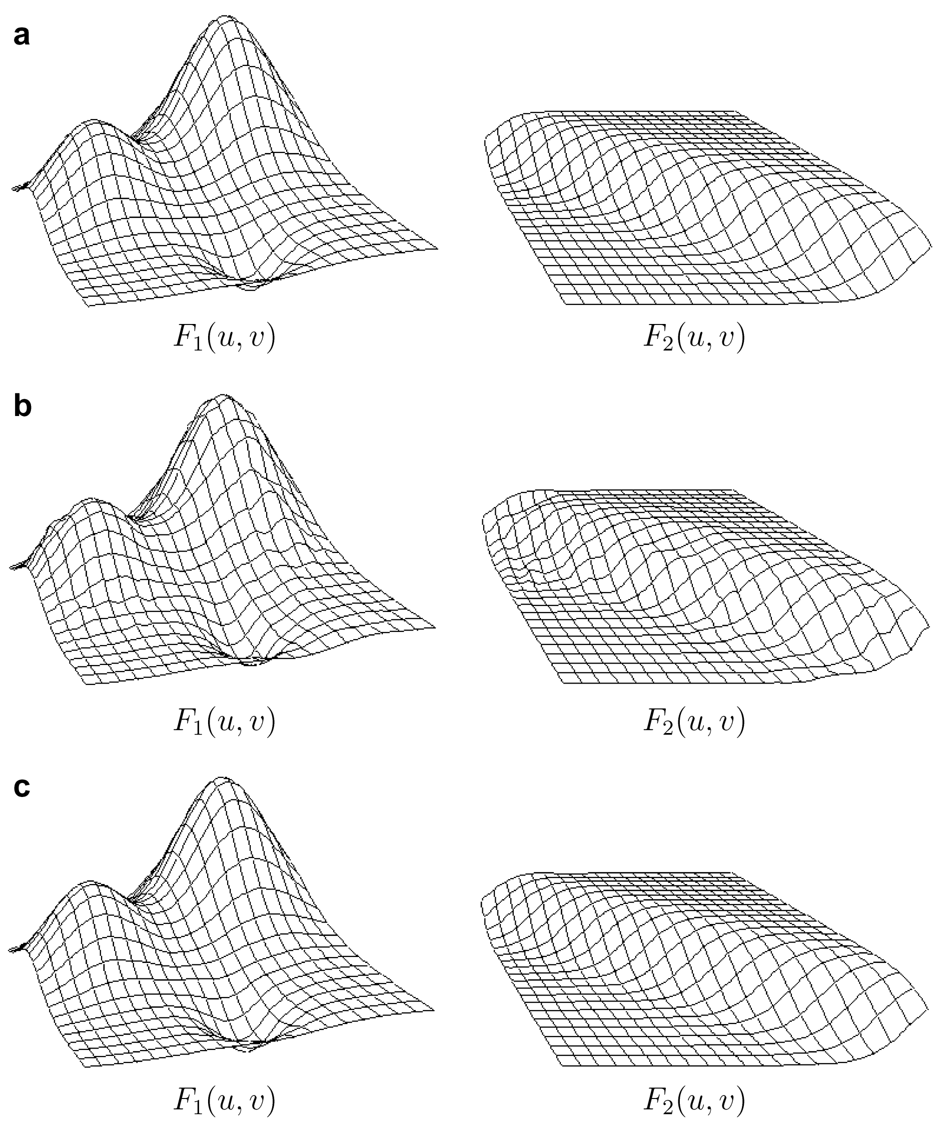

Fig. 8. Plots of $F_1(u, v)$ and $F_2(u, v)$.

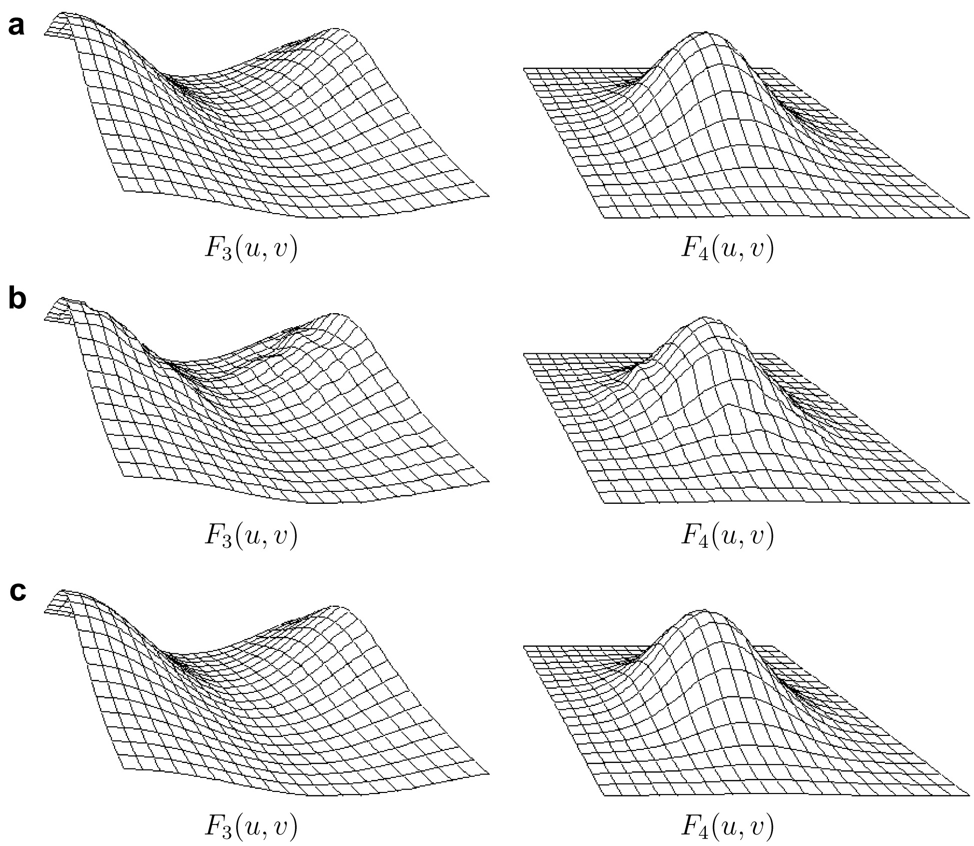

Fig. 9. Plots of $F_3(u,v)$ and $F_4(u,v)$.

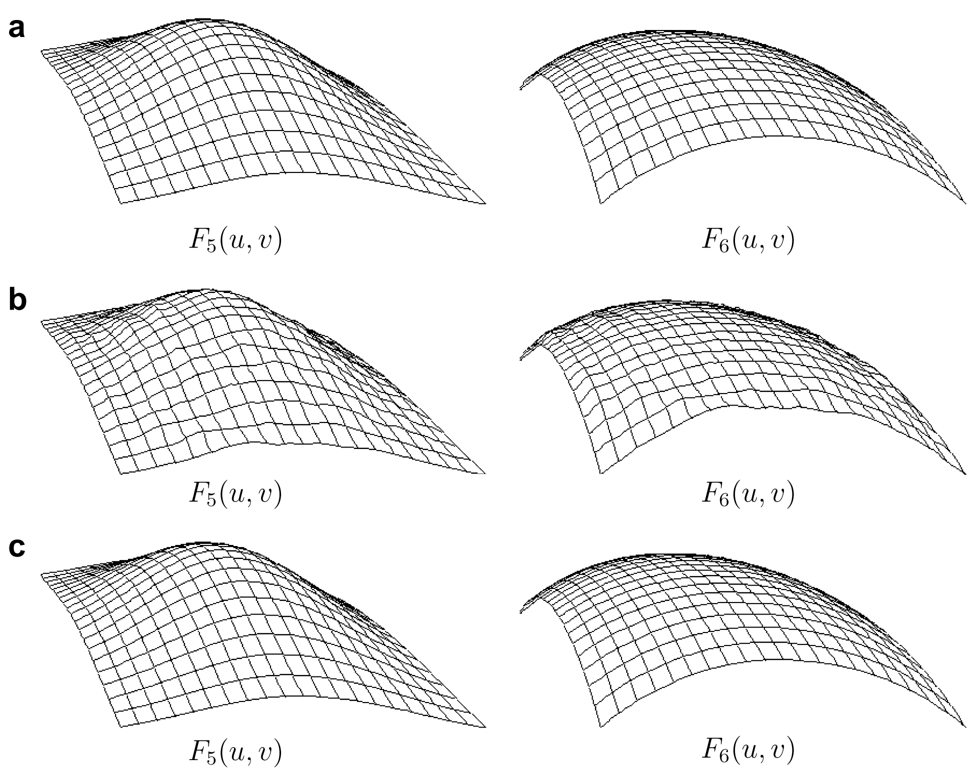

Fig. 10. Plots of $F_5(u,v)$ and $F_6(u,v)$.

tions, while the surface (c) $F_i(u,v)$ is produced with the method [12] by the boundary curves, and the cross-boundary conditions which are redefined by the new method. We can see that the surfaces (a) $F_i(u,v)$ and (b) $F_i(u,v)$ visually have no difference.

We have done the comparison by replacing the method [12] with Nielson's method [5], the test results are that: (1) for these interpolation conditions, Nielson's method cannot produce better surfaces; (2) when the new method is applied on the Nielson's method, the quality of the surfaces constructed is improved.

6. Conclusion

A new method has been proposed, which uses functional triangular patch construction method to construct parametric triangular patches. Our study has shown that the new method improves previous methods in both surface shape and surface quality, which is verified by examining Mach band effect and highlight line models of the resulting surface patches. The key in achieving the improvement is a technique to define the cross-boundary conditions. The resulting cross-boundary conditions have not only suitable magnitudes but also suitable directions.

With the new method, one can directly apply any of the classic functional triangular patch construction methods to construct a C^1 parametric triangular patch to smoothly connect three surface patches. The new method preserves precision of the classic methods. If the applied classic method has a precision of polynomials of degree n, then the constructed parametric triangle patches have a precision of parametric polynomials of degree n.

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